Notes on Spectral ~~Propertys~~ Properties of Random Graphs

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1 Introduction

In these notes, we explore the spectral properties of random graphs, with particular emphasis on the ~~eighenvalue~~ eigenvalue distribution of their adjacency matrices. The study of random graphs was initiated by Erdős and ~~Renyi~~ Rényi [1], and has since become a central topic in discrete mathematics and theoretical computer ~~seince~~science.

2 Preliminaries

Definition 1. *An ~~Erdős-Rényi~~ Erdős-Rényi random graph* $G(n, p)$ *is a graph on* n *vertices where each ~~posible edge~~ possible edge* $\{u, v\}$ *is included with probability* p *~~independent~~ , independently of all other edges.*

Let ~~A be the adjacency matrix of a graph~~ G be a graph on n vertices. Its adjacency matrix $A = A(G)$ is an $n \times n$ matrix where $A_{uv} = 1$ if $\{u, v\}$ is an edge in G~~. The eigenvalues of~~ , and $A_{uv} = 0$ otherwise. Since A ~~are denoted~~ is symmetric, its eigenvalues are real and are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

3 Spectral Gap in Random Regular Graphs

For a d-regular graph G (where every vertex has degree d), it is well-known that the largest eigenvalue of its adjacency matrix is $\lambda_1 = d$. The spectral gap, defined as $d - \lambda_2$, plays a crucial role in determining the expansion properties of the graph.

Theorem 1 (Alon-Bopanna)**.** *For any sequence of d-regular ~~graph~~ graphs G_n on n vertices whose girth tends to infinity as $n \to \infty$, we have*

$$\lambda_2(G_n) \geq 2\sqrt{d-1} - o(1) \tag{1}$$

where the $o(1)$ ~~tends to zero as the diameter of the graph tends to infinity.~~ term vanishes as $n \to \infty$.

~~However~~Conversely, for random d-regular graphs, ~~we can establish a stronger result~~

~~For any~~ Friedman proved a matching upper bound:

Theorem 2 (Friedman's Theorem [2])**.** *For any fixed $d \geq 3$ and any $\epsilon > 0$, a random d-regular graph G ~~satisfies~~ on n vertices satisfies*

$$\max(|\lambda_2(G)|, |\lambda_n(G)|) \leq 2\sqrt{d-1} + \epsilon \tag{2}$$

with probability approaching 1 as $n \to \infty$.

The proof relies on the trace method and careful analysis of ~~the moments of~~ moments of non-backtracking walks.

4 Concentration of Eigenvalues

For the ~~Erdős-Rényi~~ Erdős-Rényi random graph $G(n,p)$ with ~~$p = \frac{d}{n}$~~ $p = d/n$ for some constant $d > 0$ (corresponding to the sparse regime with average degree d), the eigenvalues of the adjacency matrix ~~exhibit interesting concentration phenomena.~~ concentrate near specific values.

Proposition 3. *Let A be the adjacency matrix of $G(n,p)$ with ~~$p = \frac{d}{n}$. Then,~~ $p = d/n$. The following properties hold almost surely as $n \to \infty$:*

1. *$\lambda_1 = (1 + o(1))np = (1 + o(1))d$ ~~almost surely~~ .*
2. *For $i \geq 2$, ~~$|\lambda_i| \leq (2 + o(1))\sqrt{np(1-p)} = (2 + o(1))\sqrt{d}$ almost surely~~ $|\lambda_i| \leq (2 + o(1))\sqrt{np(1-p)} = (2 + o(1))\sqrt{d(1-d/n)} = (2 + o(1))\sqrt{d}$.*

This shows that the bulk of the spectrum (eigenvalues $\lambda_2, \ldots, \lambda_n$) is concentrated in ~~a band of width $O(\sqrt{d})$~~ an interval of radius approximately $2\sqrt{d}$ around the origin, while the largest eigenvalue λ_1 is separated from this ~~band.~~ bulk.

References

[1] P. Erdős and A. ~~Rényi~~Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hung. Acad. Sci, 5(1):17–60, 1960.

[2] J. Friedman, *A proof of Alon's second eigenvalue conjecture and related problems*, Memoirs of the American Mathematical Society, 195(910), 2008.

[3] E. P. Wigner, *On the distribution of the roots of certain symmetric matrices*, Annals of Mathematics, 67(2):325–327, 1958.