# Notes on Spectral Properties of Random Graphs

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#### 1 Introduction

In these notes, we explore the spectral properties of random graphs, with particular emphasis on the eigenvalue distribution of their adjacency matrices. The study of random graphs was initiated by Erdős and Rényi [1], and has since become a central topic in discrete mathematics and theoretical computer science.

#### 2 Preliminaries

**Definition 1.** An Erdős-Rényi random graph G(n, p) is a graph on n vertices where each possible edge  $\{u, v\}$  is included with probability pindependent, independently of all other edges.

Let A be the adjacency matrix of a graph G be a graph on n vertices. Its adjacency matrix A = A(G) is an  $n \times n$  matrix where  $A_{uv} = 1$  if  $\{u, v\}$  is an edge in G. The eigenvalues of , and  $A_{uv} = 0$  otherwise. Since A are denoted is symmetric, its eigenvalues are real and are denoted by  $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ .

### 3 Spectral Gap in Random Regular Graphs

For a *d*-regular graph G (where every vertex has degree d), it is well-known that the largest eigenvalue of its adjacency matrix is  $\lambda_1 = d$ . The spectral gap, defined as  $d - \lambda_2$ , plays a crucial role in determining the expansion properties of the graph.

**Theorem 1** (Alon-Bopanna). For any sequence of d-regular graph-graphs  $G_n$  on n vertices whose girth tends to infinity as  $n \to \infty$ , we have

$$\lambda_2(G_n) \ge 2\sqrt{d-1} - o(1) \tag{1}$$

where the o(1) term vanishes as  $n \to \infty$  for sequences of graphs whose girth tends to infinity.

However<u>Conversely</u>, for random *d*-regular graphs, we can establish a stronger result.

For any Friedman proved a matching upper bound:

**Theorem 2** (Friedman's Theorem [2]). For any fixed  $d \ge 3$  and any  $\epsilon > 0$ , a random d-regular graph G satisfies on n vertices satisfies

$$\max(|\lambda_2(G)|, |\lambda_n(G)|) \le 2\sqrt{d-1} + \epsilon \tag{2}$$

with probability approaching 1 as  $n \to \infty$ .

The proof relies on the trace method and careful analysis of moments - of non-backtracking walks.

#### 4 Concentration of Eigenvalues

For the Erdős-Rényi random graph G(n, p) with  $p = \frac{d}{n} p = d/n$  for some constant d > 0 (corresponding to the sparse regime with average degree d), the eigenvalues of the adjacency matrix exhibit interesting concentration phenomena. concentrate near specific values.

**Proposition 3.** Let A be the adjacency matrix of G(n, p) with  $p = \frac{d}{n}$ . Then, p = d/n. The following properties hold almost surely as  $n \to \infty$ :

- 2. For  $i \ge 2$ ,  $|\lambda_i| \le (2+o(1))\sqrt{np(1-p)} = (2+o(1))\sqrt{d(1-d/n)} = (2+o(1))\sqrt{dalmost surely}$ .

This shows that the bulk of the spectrum (eigenvalues  $\lambda_2, \ldots, \lambda_n$ ) is concentrated in a band of width  $O(\sqrt{d})$  an interval of radius approximately  $2\sqrt{d}$  around the origin, while the largest eigenvalue  $\lambda_1$  is separated from this band. bulk.

## References

- P. Erdős and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hung. Acad. Sci, 5(1):17–60, 1960.
- [2] J. Friedman, A proof of Alon's second eigenvalue conjecture and related problems, Memoirs of the American Mathematical Society, 195(910), 2008.
- [3] E. P. Wigner, On the distribution of the roots of certain symmetric matrices, Annals of Mathematics, 67(2):325–327, 1958.